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Weber's Problem with Attraction and Repulsion under Polyhedral Gauges

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Abstract. Given a finite set of points in the plane and a forbidden region $\mathcal{R}$, we want to find a point $X \notin int(\mathcal{R})$, such that the weighted sum to all given points is minimized. This location problem is a variant of the well-known Weber Problem, where we measure the distance by polyhedral gauges and allow each of the weights to be positive **or** negative. The unit ball of a polyhedral gauge may be any convex polyhedron containing the origin. This large class of distance functions allows very general (practical) settings – such as asymmetry – to be modeled. Each given point is allowed to have its own gauge and the forbidden region $\mathcal{R}$ enables us to include negative information in the model. Additionally the use of negative and positive weights allows to include the level of attraction or dislikeness of a new facility. Polynomial algorithms and structural properties for this global optimization problem (d.c. objective function and a non-convex feasible set) based on combinatorial and geometrical methods are presented.

Key words: Location Theory, global optimization, discretization, geometrical algorithms

1. Introduction

We denote with $Ex = \{Ex_1, \ldots, Ex_M\}$ the given finite set of existing facilities, represented by points in $\mathbb{R}^2$, where $Ex_m = (a_m, b_m)$ for $m \in \mathcal{M} := \{1, \ldots, M\}$.

The new facility (or more precisely its co-ordinates) we want to find is denoted by X. Every existing facility Ex_m has assigned a positive or negative value not equal to zero denoted by w_m, for all $m \in \mathcal{M}$.

Since we plan to find an optimal location for the new facility, we have to have a criterion, which tells us something about the quality of the solution.
We will be concerned with:

$$\sum_{m \in \mathcal{M}} w_m d_m(Ex_m, X) =: f(X) \ .$$

The corresponding optimization problem is

$$\min_{X \in \mathcal{F} \subseteq \mathbb{R}^2} f(X) \ ,$$

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which in literature is called Weber- or Minisum- or continuous Median-Problem (see [7] or [13]) with attraction and repulsion.

In the definition of $f(X)$, $d_m(Ex_m, X)$ means the distance between the points Ex_m and X, where we allow different kinds of distances for different existing facilities.
A possible interpretation of the weights w_m is as follows:

- $w_m > 0$ can be interpreted as transportation cost per distance unit, that means the greater the distance between Ex_m and the new location is, the higher are the costs; in other words: a existing facility with weight $w_m > 0$ attracts the new location because of the increasing costs with increasing distance.
- $w_m < 0$ may be a measure for the disapproval of neighbors, who don't like the new location in their neighborhood, that means these costs decrease with increasing distance, in other words: an existing facility with $w_m < 0$ repulses the new location (the objective function is the better the farer away the new facility is located).
- $w_m = 0$ means that the costs do not depend on the location of the new facility, so we can neglect the existing facilities with $w_m = 0$ (therefore we can assume $w_m \gtrless 0$).

The set $\mathcal{F} \subseteq \mathbb{R}^2$ over which we minimize is called the feasible region.

In the classical Weber- or Minisum- or continuous Median-Problem we have $\mathcal{F} = \mathbb{R}^2$. The set of globally optimal solutions to this optimization problem (with $\mathcal{F} = \mathbb{R}^2$) is denoted $\mathcal{X}^*(f)$. The set of locally optimal solutions is denoted $\mathcal{X}^{loc}(f)$.

If we introduce a connected set $\mathcal{R} \subseteq \mathbb{R}^2$ as a forbidden region, where it is not permitted to place a new facility, we have $\mathcal{F} = \mathbb{R}^2 \backslash int(\mathcal{R})$.

Now optimizing f becomes complicated, since $\mathcal{F}$ need not be convex any more. But from a practical point of view it is a necessary extension of the classical location model, since forbidden regions appear everywhere: nature reserves, lakes, places we don't possess, etc.

These problems are called restricted location problems and have been studied for example in [16], [8], [1], [9], and [17]. The set of globally optimal solutions to these restricted location problems is denoted $\mathcal{X}^*_{\mathcal{R}}(f)$, to emphasize the influence of the forbidden region $\mathcal{R}$.

In the following we assume $\mathcal{X}^*(f) \subseteq int(\mathcal{R})$ to avoid the trivial case where $\mathcal{X}^*(f) \cap \mathcal{X}^*_{\mathcal{R}}(f) \neq \emptyset$, i.e. one of the optimal solutions of the unrestricted problem is also a solution of the restricted one.

Only a few papers have looked at extensions to the Weber problem as a global optimization problem (see [3], [18], [2], [19], [14] and [20]).

The existing papers can be roughly divided into two categories. Papers in the first category ([3], [18]) focus on structural results for general settings. The main topic is to find conditions for the finiteness of the globally optimal solution. The papers in the second category ([2], [19], [14] and [20]) apply general d.c. optimization techniques to develop iterative algorithms with a good convergence rate. The benefit of these approaches is that quite general cost functions can be taken into account.

In contrast to all these papers, this text focusses on the combinatorial structure of the Weber problem which can be established for a broad class of distance functions. This combinatorial structure allows us to show discretization results and therefore combinatorial techniques can be used instead of convergence results. Additionally, we also look at the Weber problem with different types of forbidden regions, which means that we solve d.c. problems over a non-convex feasible region. To our best knowledge nobody has discussed the Weber Problem with positive and negative weights and forbidden regions yet.

The rest of the paper is organized as follows: In the second section we will introduce a classification scheme for location problems, state some basic properties and define the class of distance functions we will investigate in this paper. Section 3 discusses the principal techniques we will use to solve unrestricted Weber Problems with positive and negative weights. In Section 4 we present an efficient algorithm to solve restricted Weber Problems for a broad class of distance functions with convex and non-convex forbidden regions. Section 5 shows how for a specific class of distance functions better complexity bounds can be obtained. The paper ends with some conclusions.

2. Definitions and Basic Concepts

2.1. A CLASSIFICATION SCHEME FOR LOCATION PROBLEMS

As one notices, the nomenclature for location problems is not unique. Therefore, we introduce in the following a classification scheme for location problems, which should help to get an overview over the manifold area of location problems.

We use a scheme which is analogous to the one introduced successfully in scheduling theory. The presented scheme for location problems was developed in [10], [11] and [12]. We have the following five position classification

$$pos1/pos2/pos3/pos4/pos5\ ,$$

where the meaning of each position is explained in the following table:

If we do not make any special assumptions in a position, we indicate this by a $\bullet$. For example, a $\bullet$ in Position 4 means that we are talking about any distance function. A $\bullet$ in Position 3 indicates that we have $w_m \geq 0$, which is the usual assumption in location theory.

Using this classification the Weber- or Minisum- or continuous Median-Problem with attraction and repulsion is written as $1/P/w_m \gtrless 0/\bullet/\sum$. The restricted Weber problem with attraction and repulsion is written as $1/P/\mathcal{R}, w_m \gtrless 0/\bullet/\sum$.

Position	Meaning	Usage (Examples)
1	number of new facilities	
2	type of problem	**P** planar location problem **D** discrete location problem **G** location problem on a network
3	special assumptions and restrictions	$w_m = 1$ all weights are equal $\mathcal{R}$ a forbidden region
4	type of distance function	l_1 Manhattan metric γ a general gauge
5	type of objective function	$\sum$ Median problem max Center problem

2.2. About the distance functions

Let B be a compact convex set in $\mathbb{R}^2$ containing the origin in its interior and let $X \in \mathbb{R}^2$. The gauge of X with respect to B is then defined as

$$\gamma(X) := \inf\{\lambda > 0 \ : \ X \in \lambda B\} \ .$$

This definition dates back to [15].

γ is a convex function and if B is symmetric with respect to the origin γ defines a norm and B is the corresponding unit ball.

Now we can define the distance from X to Y (remember that we do not necessarily have symmetry) by

$$d(X,Y) := \gamma(Y - X) \ .$$

In this paper we allow each existing facility Ex_m, $m \in \mathcal{M}$ to have its own unit ball B_m, $m \in \mathcal{M}$, being a convex polytope with extreme points $Ext(B_m) := \{e_1^m, \ldots, e_{G(m)}^m\}$ and corresponding gauge γ_m, $m \in \mathcal{M}$. In this case we can compute $\gamma_m(X)$ as

$$\gamma_m(X) = \min\left\{\sum_{g=1}^{G(m)} \lambda_g \ : \ X = \sum_{g=1}^{G(m)} \lambda_g e_g^m \ , \ \lambda_g \geq 0\right\}$$

(see [6]).

Gauges with a polyhedral unit ball are called polyhedral gauges. If, additionally, the unit ball B_m is symmetric, γ_m is called a block norm.

Let $d_1^m, \ldots, d_{G(m)}^m$ be the halflines defined by the endpoint 0 and $e_1^m, \ldots, e_{G(m)}$, $m \in \mathcal{M}$. The set of halflines $d_1, \ldots, d_{G(m)}$ is called fundamental directions. By setting $d_{G(m)+1}^m := d_1^m$ we define Γ_g^m as the cone generated by d_g^m and d_{g+1}^m. The translated set $X + B_m$ is denoted B_m^X.

Two well-known block norms belong to the class of l_p-norms: The l_1 or rectilinear norm and the l_∞ or maximum norm. The unit ball of the l_1 norm is the polyhedron with extreme points $\{(1,0),(0,-1),(-1,0),(0,1)\}$ and the unit ball of the l_∞ norm is the polyhedron with extreme points $\{(1,1),(1,-1),(-1,-1),(-1,1)\}$.

Since there exists a linear norm-converting map T between the two block norms l_1 and l_∞ (see [7]), we can use all algorithms which we will develop for l_1 automatically for l_∞ too.

The importance of polyhedral gauges becomes even clearer if one notes that, since a convex set can be approximated by a convex polyhedron to within any specified ε-degree of tolerance (see [21]), the following results hold.

THEOREM 2.1. *The class of polyhedral gauges is dense in the set of all gauges.*

COROLLARY 2.2 (see [22]). *The class of block norms is dense in the set of all norms.*

In the following we will only look at polyhedral gauges.

2.3. About the objective function

We can reformulate the objective function f in the following way:

$$f(X) = \sum_{m \in \mathcal{M}^+} w_m \gamma_m(X - Ex_m) - \sum_{m \in \mathcal{M}^-} (-w_m)\gamma_m(X - Ex_m) ,$$

where $\mathcal{M}^+ := \{m \ : \ w_m > 0\}$ and $\mathcal{M}^- := \{m \ : \ w_m < 0\}$. This type of functions is well known in global optimization and is called d.c. functions, which stands for difference of convex functions.

If both index sets $\mathcal{M}^+$ and $\mathcal{M}^-$ are non-empty, the objective function f is neither convex nor concave, which means there may exist several local minima and we have to find out which of them is the global one.

It is clear that if the total weight of facilities belonging to $\mathcal{M}^-$ becomes too large, the minimum will be $-\infty$. This is made more precise in the following theorem:

THEOREM 2.3 (see [3]). *Let* $W := \sum_{m \in \mathcal{M}} w_m$.

Then the following holds:

- *For* $W > 0$ *the optimal location is finite,*
- *For* $W < 0$ *the optimal solution is at infinity,*
- *For* $W = 0$ *the result depends on the input data and the metric, so no general result can be formulated.*

We can also give a sufficient condition for the optimality of an existing facility:

THEOREM 2.4 (see [2]). *If a* m^* *exists with* $w_{m^*} \geq \sum_{\substack{m \in \mathcal{M} \\ m \neq m^*}} |w_m|$, *then* Ex_{m^*} *is the optimal location.*

As a consequence we only have to look for problems with $W \geq 0$, when we are solving $1/P/w_m \gtrless 0/\gamma_m/\sum$.

2.4. Level curves and level sets

In the following we will introduce level curves and level sets and reformulate restricted and unrestricted location problems using these concepts.

For a function h from $\mathbb{R}^2$ to $\mathbb{R}_+$ and $z \in \mathbb{R}$ the **level curve** $L_=(z)$ and the **level set** $L_\leq(z)$ is defined by

$$L_=(z) := \{X \in \mathbb{R}^2 : h(X) = z\}$$

and

$$L_\leq(z) := \{X \in \mathbb{R}^2 : h(X) \leq z\}$$

respectively.

Using level curves and level sets we can reformulate $1/P/w_m \gtrless 0/\bullet/\bullet$ and $1/P/\mathcal{R},\ w_m \gtrless 0/\bullet/\bullet$.

THEOREM 2.5.

a) *z^* is the optimal objective value of* $1/P/w_m \gtrless 0/\bullet/\bullet$
$\Leftrightarrow z^* = min\{z : L_=(z) \neq \emptyset\}$.

b) *$z^{\mathcal{F}}$ is the optimal objective value of* $1/P/\mathcal{R},\ w_m \gtrless 0/\bullet/\bullet$
$\Leftrightarrow z^{\mathcal{F}} = min\{z : L_=(z) \cap \mathcal{F} \neq \emptyset\}$.

c) *In a) and b) $L_=(z)$ can be replaced by $L_\leq(z)$.*

Using this theorem we can implement a search procedure to values of z until the optimality conditions are satisfied or any other stopping criterion terminates the procedure. However this is not very satisfactory but we will see in the following sections that level curves and level sets lead to efficient discretization procedures for $1/P/w_m \gtrless 0/\gamma_m/\sum$ and $1/P/\mathcal{R},\ w_m \gtrless 0/\gamma_m/\sum$.

3. Solving $1/P/w_m \gtrless 0/\gamma_m/\sum$

Now, we consider the unrestricted problem under polyhedral gauges. We do **not** assume the same gauge for every Ex_m, $m \in \mathcal{M}$.

First the structure of the level curves will be examined.

THEOREM 3.1 (see [17] or [22]).
The polyhedral gauge γ_m is linear over the cone Γ_g^m, for $g = 1, \ldots, G(m)$.

Let $\pi = (p_m)_{m \in \mathcal{M}}$ be a family of numbers such that $p_m \in \{1, \ldots, G(m)\}$ for all $m \in \mathcal{M}$ and let

$$C_\pi = \bigcap_{m \in \mathcal{M}} \left(Ex_m + \Gamma_{p_m}^m\right)$$

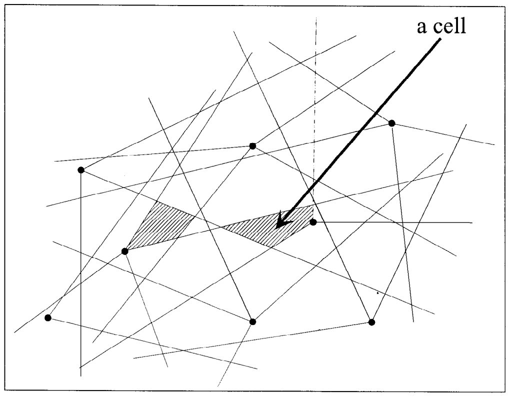

Figure 3.1. An example for the set $\mathcal{C}$.

A convex set C, with $int(C) \neq \emptyset$, is said to be a cell if there exists a family π such that $C_\pi = C$ (see [5]).

REMARK. Geometrically we obtain all cells if we draw for every $Ex_m \in Ex$ all halflines d_g, $g = 1, \ldots, G(m)$ starting at Ex_m.

The set of all cells is called $\mathcal{C}$. For an example of such a system of cells see Figure 3.1.

THEOREM 3.2. *The level curves of $f(X)$ with polyhedral gauges are linear in each $C \in \mathcal{C}$.*

Proof. With Theorem 3.1 we have $\gamma_m(X)$ is linear in each cone Γ_g^m. Since all cells are intersections of such cones, $\gamma_m(X)$ is linear in each cell $C \in \mathcal{C}$.

For $X \in C$ we can therefore write

$$f(X) = \sum_{m \in \mathcal{M}} w_m l_m (X - Ex_m) ,$$

where the l_m are linear functions in X. It follows that $f(X) = z$ is linear in C.

It is clear that by definition $\mathbb{R}^2 = \cup_{C \in \mathcal{C}}$. Now we can characterize the set of local optima for $1/P/w_m \gtrless 0/\gamma_m/\sum$. □

THEOREM 3.3.
A connected component of $\mathcal{X}^{loc}(f)$ with level z is either
- *a complete cell*
- *a facet of a cell or*
- *an extreme point of a cell.*

Proof. Follows directly from Theorem 3.2 and the linearity of the objective function f in a cell. □

COROLLARY 3.4. *For finding $\mathcal{X}^*(f)$ or $\mathcal{X}^{loc}(f)$, it suffices to look at the $O(M^2(\max_{m\in\mathcal{M}} G(m))^2)$ extreme points of all cells.*

Let

$$\mathcal{H} := \bigcup_{m\in\mathcal{M}} \left\{ \bigcup_{g=1}^{G(m)} Ex_m + d_g^m \right\} ,$$

i.e. the union of all points on halflines in the direction of all $Y \in Ext(B_{\gamma m}^{Ex_m})$ for all existing facilities. The halflines formed by $\mathcal{H}$ are called construction lines. The set of intersection points generated by $\mathcal{H}$ is denoted $\mathcal{I}$. Note that $\mathcal{I}$ equals the set of extreme points of all cells.

EXAMPLE 3.1. We are given four existing facilities $Ex_1 = (0.5, 0.5)$, $Ex_2 = (3, 9)$, $Ex_3 = (7, 3)$ and $Ex_4 = (11, 7)$. The corresponding weights are $w_1 = 4.1$, $w_2 = w_3 = -1$ and $w_4 = 2.9$. Each facility Ex_m is assigned a different gauge γ_m, defined by the extreme points of B_m, for $m = 1, \ldots, 4$, where $Ext(B_1) = \{(1, 1), (-1, 1), (0, -1)\}$, $Ext(B_2) = \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$, $Ext(B_3) = \{(0, 1), (-1, -1), (1, -1)\}$ and $Ext(B_4) = \{(0, 1), (-1, 0), (0, -1), (1, 0)\}$. To find the set of optimal locations $\mathcal{X}^*(f)$, we have to inspect all points in $\mathcal{I}$. These intersection points together with the corresponding objective value are given in the following table.

$X \in \mathcal{I}$	$(-6, 7)$	$(-2.5, 3.5)$	$(0.5, -3.5)$	$(0.5, 0.5)$	$(1, 7)$	$(3, 9)$	$(5, 7)$	$(6, 6)$
$f(X)$	36.6	58.3	30.3	37.65	49.85	34.05	31.95	60.25

$X \in \mathcal{I}$	$(7, 3)$	$(7, 5)$	$(7, 7)$	$(7, 13)$	$(11, -1)$	$(11, 1)$	$(11, 7)$	$(11, 11)$	$(11, 17)$
$f(X)$	36.95	46.25	30.25	66.25	101.45	87.45	39.45	30.65	66.65

From this table we get that the optimal solution is in this example a single point, $\mathcal{X}^*(f) = \{(7, 7)\}$ with objective value 30.25. In Figure 3.2 a graphical representation of the major part of this example is shown.

4. Solving $1/P/\mathcal{R},\ w_m \gtrless 0/\gamma_m/\sum$

4.1. PRINCIPAL TECHNIQUES

To describe a general solution procedure for the restricted problem we need to know a little bit more about the structure of the level sets of f. Remember that

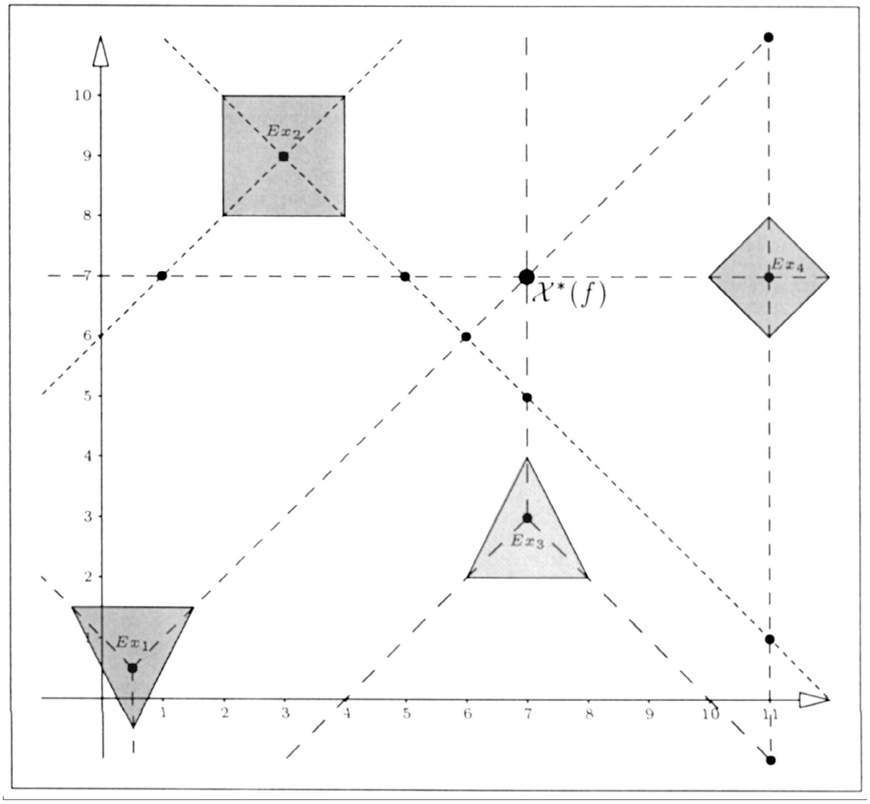

Figure 3.2. The existing facilities as well as the major part of the sets $\mathcal{H}$ and I of Example 3.1.

we exclude a connected region in $\mathbb{R}^2$ and so we have in general to optimize a non-convex objective function over a non-convex domain. From Theorem 3.2 we know that the level curves are piecewise linear independent of the value of W, the sum of all weights. For the level sets $L_{\leq}(z)$ the situation is a little bit more complicated.

THEOREM 4.1. *The level sets $L_{\leq}(z)$ for the objective function f have the following form:*

$W > 0$ *The level curves $L_{=}(z)$ are closed polygons and the corresponding level sets $L_{\leq}(z)$ are the bounded sets defined by the boundary $L_{=}(z)$.*

$W < 0$ *The level curves $L_{=}(z)$ are closed polygons and the corresponding level set $L_{\leq}(z)$ is the unbounded exterior of the level curve $L_{=}(z)$, i.e. $L_{\leq}(z) = \mathbb{R}^2 \setminus int(L^-_{\leq}(z))$, where $L^-_{\leq}$ is the level set with respect to $-f$.*

Proof.

$W > 0$ By Theorem 2.3 the optimal solution is finite and by Theorem 3.3 the structure of a local optimum is known. By the piecewise linearity of the level curves (see Theorem 3.2) and the finiteness of the local optima the level curves are closed polygons around these local optima and the level sets have to include these local optima.

$W < 0$ The validity of the statement follows by multiplying f by -1. Now we have $W > 0$ and we are in the first case. Since $-f(x) \geq z$ is equivalent to $f(x) \leq -z$ the result follows. □

REMARK. For the l_1-case, with $W = 0$, it is shown in [4] that the level curves are horizontal or vertical lines outside the convex hull of $\mathcal{I}$. ($\mathcal{I}$ is the set of intersection points defined by $\mathcal{H}$). Therefore in the case $W = 0$ the level curves need not to be closed anymore.

Now we will look at several types of forbidden regions:
First the forbidden region is assumed to be any bounded convex set, second we consider any closed polygon (not necessarily convex) and third we look at the complement of a closed polygon as the forbidden region. In all situations we distinguish between the three cases $W > 0$, $W < 0$ and $W = 0$. Notice also that we exclude the trivial case where an optimal solution for the unrestricted problem is also feasible for the restricted problem, i.e. $\mathcal{X}^*(f) \cap \mathcal{F} \neq \emptyset$.

THEOREM 4.2. *X is an optimal solution of $1/P/\mathcal{R},\ w_m \gtrless 0/\gamma_m/\sum$ $(X \in \mathcal{X}^*_{\mathcal{R}}(f))$ with $f(X) = z$ if and only if there exists a $z \in \mathbb{R}$, such that*

a) *$X \in \mathcal{X}^{loc}(f) \cap \mathcal{F}$ and*

$$z = \min\left\{ f(Y) \ : \ Y \in \mathcal{X}^{loc}(f) \cap \mathcal{F} \right\} \tag{4.1}$$

or

b)

$$L_{=}(z) \cap \partial\mathcal{R} \neq \emptyset \tag{4.2}$$

and

$$L_{\leq}(z) \subseteq \mathcal{R}\ . \tag{4.3}$$

Of course, both cases may coincide.

Proof. If Case a) is fulfilled we get the best feasible local minimum which is then of course globally optimal. For Case b) note that we can conclude from Theorem 3.2 and Theorem 4.1 that $int(L_{\leq}(z)) = L_{<}(z)$ for $z > z^*$, where z^* is the global minimum of f. Suppose now (4.2) and (4.3) hold and there is no $X \in \mathcal{X}^{loc}(f)$ satisfying (4.1). Then every level set with a smaller level than z is completely infeasible and every level set with a larger level than z is not optimal. So the cases a) and b) are sufficient for showing X to be optimal.

Now suppose we have an optimal X with $f(X) = z$ and neither Case a) nor Case b) holds. We can not have a better locally optimal solution $Y \in \mathcal{F}$, because then Case a) would hold. Since $X \in \mathcal{F}$ and also Case b) does not hold we have that $int(L_{\leq}(z)) \cap \partial\mathcal{R} \neq \emptyset$ and therefore we have feasible points on the boundary with a better objective value than X. So X can not be optimal if neither Case a) nor Case b) is fulfilled. □

Note that the proof does not only hold for the Weber objective function, but also for other objective functions, like the center objective.

4.2. $\mathcal{R}$ IS A CONVEX SET

Here we only have to consider cases with $W \geq 0$ because for $W < 0$ the optimal solution is not finite. Therefore it is not very restricting if we assume in the following that the optimal solution is finite.

Based on Theorem 4.2 the following procedure can be used to solve $1/P/\mathcal{R}$, $w_m \gtrless 0/\bullet/\sum$.

ALGORITHM 4.1. *(Level Curve Approach for Solving $1/P/\mathcal{R}, w_m \gtrless 0/\bullet/\sum$)*

1. Find level curve $L_=(z_1)$ satisfying (4.2) and (4.3).
2. Find level z_2 satisfying (4.1).
3. If $z_1 < z_2$ let $\mathcal{X}^*_{\mathcal{R}}(f) := L_=(z_1) \cap \partial\mathcal{R}$.
4. If $z_1 \geq z_2$ let $\mathcal{X}^*_{\mathcal{R}}(f) := L_=(z_2) \cap \mathcal{F}$.

Output: $\mathcal{X}^*_{\mathcal{R}}(f)$.

The level curve approach can be implemented applying a search procedure to values of z until (4.2) and (4.3) or (4.1) is satisfied or any other stopping criterion terminates the procedure. This implementation of the procedure is, however computationally unsatisfactory, since there is no finite bound on its time complexity for finding the exact solution.

On the other hand, this approach leads in the case of polyhedral gauges to efficient procedures for solving restricted location problems, as we will see in the following.

THEOREM 4.3. *Let $W \geq 0$, let $\mathcal{R}$ be a bounded convex forbidden region and let $\mathcal{X}^*(f) \cap \mathcal{F} = \emptyset$. Then there exists an optimal location $X^*_{\mathcal{R}} \in \mathcal{X}^*_{\mathcal{R}}(f)$ with $X^*_{\mathcal{R}} \in \mathcal{H} \cap \partial\mathcal{R}$ or $X^*_{\mathcal{R}}$ is the best local minimum in $\mathcal{F}$.*

The first part of the proof is analogous to Theorem 5.2 in [8] and the second part was shown in Theorem 4.2.

So we get the following idea for an algorithm:

Solve the problem with the algorithm for the unrestricted problem; if $\mathcal{X}^*(f) \cap \mathcal{F} \neq \emptyset \rightarrow$ Stop.

For $\mathcal{X}^*(f) \cap \mathcal{F} = \emptyset$ we have to determine all feasible local minima and the intersection points of $\partial\mathcal{R}$ with the construction lines. By comparison of the objective values at these points we find the best feasible solution. If all optimal locations should be determined and the optimum is at an intersection point $\mathcal{H}$ with $\partial\mathcal{R}$, we have to compute the level curve and determine the intersection of the level curve with the boundary of $\mathcal{R}$.

More formally this reads as

ALGORITHM 4.2. *(Construction Line Algorithm for the* $1/P/\mathcal{R}, w_m \gtrless 0/\gamma_m/\sum$*)*

1. if $\mathcal{X}^*(f) \cap \mathcal{F} \neq \emptyset$ then $\mathcal{X}^*_{\mathcal{R}}(f) := \mathcal{X}^*(f) \cap \mathcal{F} \rightarrow$ Stop.
2. Compute $\mathcal{H}$.
3. Determine $\{X_1^*, \ldots, X_L^*\} = \mathcal{I} \cap \mathcal{F}$.
4. Determine $\{Y_1, \ldots, Y_K\} = \mathcal{H} \cap \partial\mathcal{R}$.
5. Let $X^*_{\mathcal{R}} \in argmin\{f(Y_1), \ldots, f(Y_K), f(X_1^*), \ldots, f(X_L^*)\}$ and let L be the level curve through $X^*_{\mathcal{R}}$ if $X^*_{\mathcal{R}} \notin \mathcal{I} \cap \mathcal{R}$.

Output: if $X^*_{\mathcal{R}} \notin \mathcal{I} \cap \mathcal{R}$
then $\mathcal{X}^*_{\mathcal{R}}(f) := L \cap \partial\mathcal{R}$
else $\mathcal{X}^*_{\mathcal{R}}(f) := L_=(f(X^*_{\mathcal{R}})) \cap \mathcal{F}$.

REMARK. By Theorem 3.3 we can determine all local optima by inspecting all extreme points of all $C \in \mathcal{C}$ and therefore Step 3 of Algorithm 4.2 is correct.

We have not more than $O(M^2(\max_{m\in\mathcal{M}} G(m))^2)$ local optima (see Corollary 3.4) and not more than $O(M \max_{m\in\mathcal{M}} G(m))$ intersection points. The evaluation of the objective function takes $O(M(\max_{m\in\mathcal{M}} G(m))$. Therefore the algorithm has a complexity of $O(M^3(\max_{m\in\mathcal{M}} G(m))^3)$.

EXAMPLE 4.1. We use the same input data as in Example 3.1. Additionally we are given $\mathcal{R} := [4, 9] \times [4.5, 8.5]$. The intersection points $\mathcal{H} \cap \partial\mathcal{R}$ together with the corresponding objective value are given in the following table.

$X \in \mathcal{H} \cap \partial\mathcal{R}$	$(4, 7)$	$(4.5, 4.5)$	$(7, 4.5)$	$(7.5, 4.5)$	$(9, 7)$	$(8.5, 8.5)$	$(7, 8.5)$	$(4, 8)$
$f(X)$	34.95	31.5	49.75	51.4	34.85	30.4	39.25	41.95

From this table and the table in Example 3.1 reporting the objective values for all points in $\mathcal{I}$ we get that the optimal solution is in this example a single point, $\mathcal{X}^*_{\mathcal{R}}(f) = \{(0.5, 0.5) = Ex_1\}$ with objective value 30.3. In Figure 4.1 a graphical representation of this example is shown.

If we change the location of Ex_1 to $(2, 2)$ the optimal solution of the modified restricted location problem is $(8.5, 8.5)$ with objective value 24.25 on the boundary of $\mathcal{R}$.

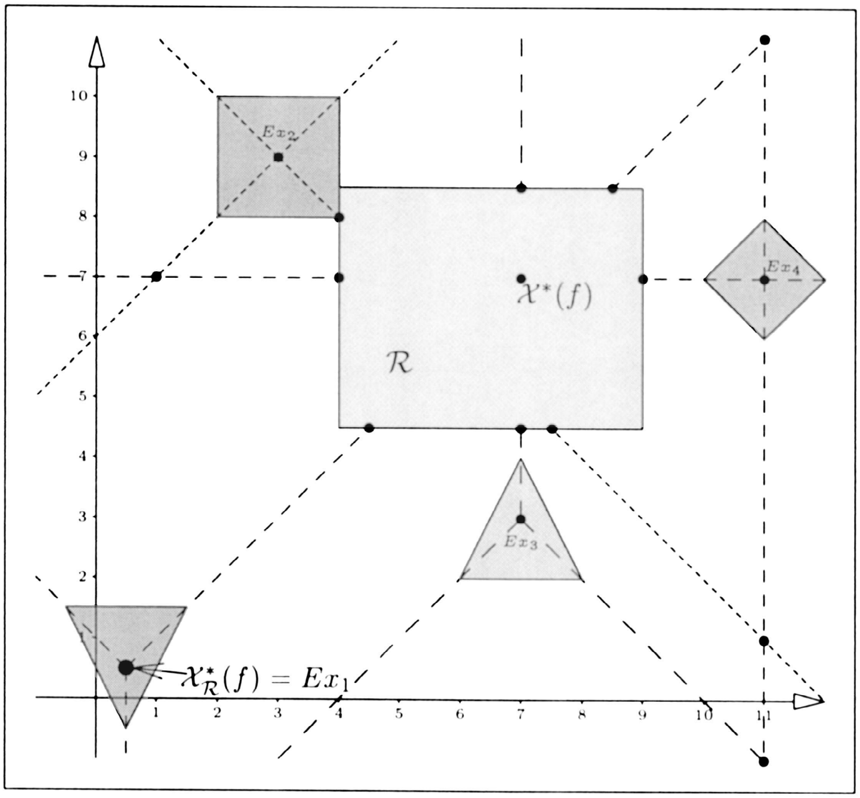

Figure 4.1. Illustration of Example 4.1.

4.3. $\mathcal{R}$ IS A BOUNDED POLYGONAL REGION

Let $\mathcal{R} = \mathcal{P}, \mathcal{P}$ any polygon, i.e. not necessarily convex. In this case we have to extend the candidate set by the set of vertices of the polygon P because of the following result:

THEOREM 4.4. *Let $W \geq 0$ (with finite optimal solution for f) and $\mathcal{R} = \mathcal{P}$ any polygon with vertex set $\mathcal{V}(\mathcal{P}) = \{V_1, \ldots, V_N\}$ and let $\mathcal{X}^*(f) \cap \mathcal{F} = \emptyset$.*
Then there exists an optimal location $X^*_{\mathcal{R}} \in \mathcal{X}^*_{\mathcal{R}}(f)$ with $X^*_{\mathcal{R}} \in \mathcal{H} \cap \partial\mathcal{R}$ or $X^*_{\mathcal{R}} \in \mathcal{V}(\mathcal{P})$ or $X^*_{\mathcal{R}}$ is the best local minimizer in $\mathcal{F}$.

Proof. Let $X^*_{\mathcal{R}} \in \partial\mathcal{R} \cap L_=(z^*_{\mathcal{R}})$ (condition for an optimal location, which is not a local optimum in $\mathcal{F}$ according to Theorem 4.2), $C \in \mathcal{C}$ with $X^*_{\mathcal{R}} \in C$, $L_C := L_=(z^*_{\mathcal{R}}) \cap C$ and $\mathcal{P}_{V_i V_{i+1}}$ defined as the segment between V_i and V_{i+1} (facet of the polygon)
Then we have the following possibilities for $X^*_{\mathcal{R}}$:

Case 1 (see Figure 4.2a) The slope changes at $X^*_{\mathcal{R}}$, so $X^*_{\mathcal{R}} \in \mathcal{H}$.

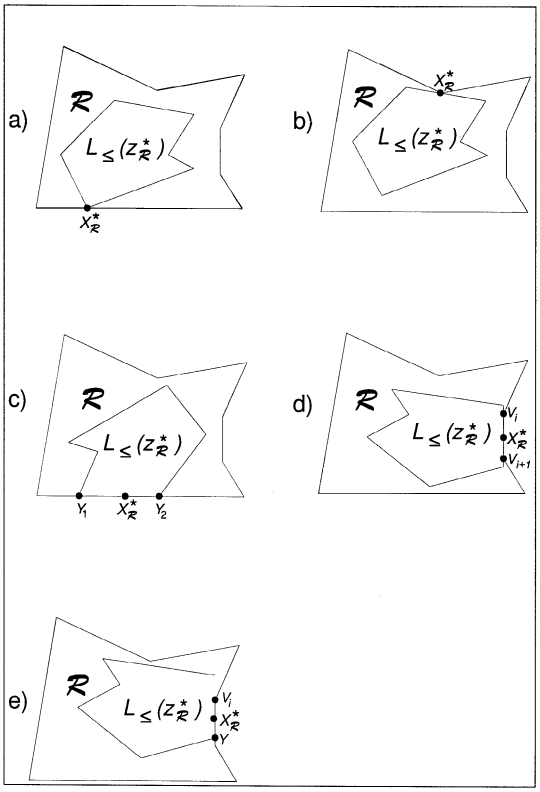

Figure 4.2. Illustrations for the different cases in the proof of Theorem 4.4

Case 2 (see Figure 4.2b) $X^*_{\mathcal{R}}$ is a vertex of $\mathcal{P}$.

Case 3 (see Figure 4.2c) L_C is a supporting line on $\mathcal{P}$ in $X^*_{\mathcal{R}}$ and $L_C \subseteq \mathcal{P}_{V_iV_{i+1}}$. Then there exist two points Y_1 and $Y_2 \in L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}$, where the slope changes, so we can replace $X^*_{\mathcal{R}}$ by Y_1 or Y_2.

Case 4 (see Figure 4.2d) L_C is a supporting line on $\mathcal{P}$ in $X^*_{\mathcal{R}}$ and $L_C \supseteq \mathcal{P}_{V_iV_{i+1}}$. Then $X^*_{\mathcal{R}}$ can be replaced by V_i or V_{i+1}.

Case 5 (see Figure 4.2e) L_C is a supporting line on $\mathcal{P}$ in $X^*_{\mathcal{R}}$, but neither $L_C \subseteq \mathcal{P}_{V_iV_{i+1}}$ nor $L_C \supseteq \mathcal{P}_{V_iV_{i+1}}$ (i.e. L_C and $\mathcal{P}_{V_iV_{i+1}}$ overlap). Then there exists a vertex $V_i \in L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}$ and a point $Y \in L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}$, where the slope changes, so we can replace $X^*_{\mathcal{R}}$ in this case too. □

4.4. $\mathcal{R}$ IS THE COMPLEMENT OF A CLOSED POLYGONAL REGION

THEOREM 4.5. *Let $\mathcal{R} := \mathbb{R}^2 \setminus \mathcal{P}$ with $\mathcal{P}$ being a polygon with vertex set $\mathcal{V}(\mathcal{P}) = \{V_1, \ldots, V_N\}$, i.e. we have a feasible region $\mathcal{F} = \mathcal{P}$ for the new location. Further $\mathcal{X}^*(f) \cap \mathcal{F} = \emptyset$.*

- *For $W \geq 0$ and if the optimal solution of the unrestricted problem is finite the following holds: There exists an optimal location $X^*_{\mathcal{R}} \in \mathcal{X}^*_{\mathcal{R}}(f)$ with $X^*_{\mathcal{R}} \in \mathcal{H} \cap \partial\mathcal{R}$ or $X^*_{\mathcal{R}} \in \mathcal{V}(\mathcal{P})$ or $X^*_{\mathcal{R}}$ is a local minimizer in $\mathcal{P}$.*
- *For $W \leq 0$ and if the optimal solution of the unrestricted problem is not finite we have: There exists an optimal location $X^*_{\mathcal{R}} \in \mathcal{X}^*_{\mathcal{R}}(f)$ with $X^*_{\mathcal{R}} \in \mathcal{H} \cap \partial\mathcal{R}$ or $X^*_{\mathcal{R}} \in \mathcal{V}(\mathcal{P})$ or $X^*_{\mathcal{R}}$ is a local minimizer in $\mathcal{P}$.*

The proof is analogous to the proof of Theorem 4.4 and is therefore omitted here.

REMARK. We also had to consider the case $W < 0$, because the solution at infinity is not feasible.

5. The Rectilinear Case

In this section we will show that for special distance measures better algorithms can be derived. Therefore we assume in the following that $\gamma_m = l_1$, for all $m \in \mathcal{M}$.

5.1. Solving $1/P/w_m \gtrless 0/l_1/\sum$

Now we consider as metric the rectangular distance, so we have $d_m = l_1$ for all $m \in \mathcal{M}$. In this case the construction lines $\mathcal{H}$ are horizontal and vertical lines through the co-ordinates of the existing facilities. So we get a decomposition of $\mathbb{R}^2$ in rectangles. For the objective function we get:

$$\begin{aligned} f(X) &= \sum_{m\in\mathcal{M}} w_m(|a_m - x_1| + |b_m - x_2|) \\ &= \underbrace{\sum_{m\in\mathcal{M}} w_m|a_m - x_1|}_{:= f_a(x_1)} + \underbrace{\sum_{m\in\mathcal{M}} w_m|b_m - x_2|}_{:= f_b(x_2)} \end{aligned} \tag{5.1}$$

One can see that the original problem can be divided in two equal subproblems, which can be solved independently:

$$f_\nu(x) = \sum_{m\in\mathcal{M}} w_m|\nu_m - x|, \text{where } \nu \in \{a, b\}\ .$$

Assume without loss of generality $\nu_1 < \nu_2 < \ldots < \nu_M$
The first step is to remove the absolute values, so we determine $p(x) := \max\{q|\nu_q \leq x, q \in \mathcal{M}\}$ and reformulate the objective function:

$$\begin{aligned} f_\nu(x) &= \sum_{m=1}^{p(x)} w_m(x - \nu_m) + \sum_{m=p(x)+1}^{M} w_m(\nu_m - x) \\ &= \left(\sum_{m=1}^{p(x)} w_m - \sum_{m=p(x)+1}^{M} w_m\right) x - \sum_{m=1}^{p(x)} w_m\nu_m + \sum_{m=p(x)+1}^{M} w_m\nu_m \end{aligned} \tag{5.2}$$

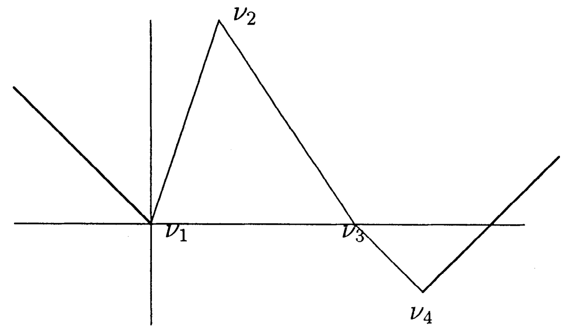

Figure 5.1. Example for $f_\nu(x)$ with 4 existing facilities

The objective function is a piecewise linear function, where the slope changes only at ν_m (the slope is constant between ν_{m-1} and ν_m). Therefore, if the minimum is finite, it can either be at a co-ordinate of an existing facility or — if the slope is 0 between ν_{m-1} and ν_m — in the whole interval between two successive co-ordinates of existing facilities.

To make this result clear, we look at the following sketch of $f_\nu(x)$ with 4 existing facilities (Figure 5.1).

Here we can see, that the derivative from the left and the right are important.

Derivatives:

Derivative from the right: $f_\nu^+(\nu_m) = \sum_{i=1}^{m} w_i - \sum_{i=m+1}^{M} w_i$

Derivative from the left: $f_\nu^-(\nu_m) = \sum_{i=1}^{m-1} w_i - \sum_{i=m}^{M} w_i$

It follows:

- $f_\nu^+(\nu_{m-1}) = f_\nu^-(\nu_m)$
- $f_\nu^-(\nu_1) = -\sum_{i=1}^M w_i = -W$
- $f_\nu^+(\nu_M) = \sum_{i=1}^M w_i = W$

THEOREM 5.1. *We have only the following two possibilities for the set of minimizers $\mathcal{X}^*(f_\nu)$:*

a) $f_\nu^-(\nu_{m^*}) < 0$ *and* $f_\nu^+(\nu_{m^*}) > 0$. *Then* $\mathcal{X}^{loc}(f_\nu) = \nu_{m^*}$ *with* $w_{m^*} > 0$.

b) $f_\nu^-(\nu_{m^*}) < 0$, $f_\nu^+(\nu_{m^*}) = 0$ *and* $f_\nu^+(\tilde{\nu}_{m^*+1}) > 0$. *Then* $\mathcal{X}^{loc}(f_\nu) = [\nu_{m^*}, \nu_{m^*+1}]$ *with* $w_{m^*}, w_{m^*+1} > 0$

Proof.

a) $$f_\nu^-(\nu_{m^*}) < 0 \Leftrightarrow \left(\sum_{i=1}^{m^*-1} w_i - \sum_{i=m^*}^{M} w_i\right) < 0 \Leftrightarrow \left(-\sum_{i=1}^{m^*-1} w_i + \sum_{i=m^*}^{M} w_i\right) > 0 \tag{1}$$

$$f_\nu^+(\nu_{m^*}) > 0 \Leftrightarrow \left(\sum_{i=1}^{m^*} w_i - \sum_{i=m^*+1}^{M} w_i\right) > 0 \tag{2a}$$

(1) + (2a) yields $w_{m^*} + w_{m^*} > 0 \Leftrightarrow w_{m^*} > 0$

b) $x \in [\nu_{m^*}, \nu_{m^*+1}]$

$$f_\nu^+(\nu_{m^*}) = 0 \Leftrightarrow \left(\sum_{i=1}^{m^*} w_i - \sum_{i=m^*+1}^{M} w_i\right) = 0 \tag{2b}$$

$$f_\nu^+(\nu_{m^*+1}) > 0 \Leftrightarrow \left(\sum_{i=1}^{m^*+1} w_i - \sum_{i=m^*+2}^{M} w_i\right) > 0 \tag{3}$$

(1) + (2b) yields $w_{m^*} + w_{m^*} > 0 \Leftrightarrow w_{m^*} > 0$
(3) − (2b) yields $w_{m^*+1} + w_{m^*+1} > 0 \Leftrightarrow w_{m^*+1} > 0$ □

THEOREM 5.2.
a) ν_{m^*} *is a local minimizer* $\Leftrightarrow$ $f_\nu^-(\nu_{m^*}) < 0$ and $w_{m^*} > \frac{1}{2} f_\nu^-(\nu_{m^*})$

b) $[\nu_{m^*}, \nu_{m^*+1}]$ *is a local minimizer* $\Leftrightarrow$ $f_\nu^-(\nu_{m^*}) < 0$, $w_{m^*} = \frac{1}{2} f_\nu^-(\nu_{m^*})$ *and* $w_{m^*+1} > 0$

Proof. Using the preceding results the following holds:

$$f_\nu^-(\nu_{m^*}) = \sum_{i=1}^{m^*-1} w_i - \sum_{i=m^*}^{M} w_i$$
$$f_\nu^+(\nu_{m^*}) = \sum_{i=1}^{m^*} w_i - \sum_{i=m^*+1}^{M} w_i$$

So we get:

$$\begin{aligned} f_\nu^+(\nu_{m^*}) - f_\nu^-(\nu_{m^*}) &= w_{m^*} + w_{m^*} = 2w_{m^*} \Leftrightarrow f_\nu^+(\nu_{m^*}) \\ &= f_\nu^-(\nu_{m^*}) + 2w_{m^*} \end{aligned}$$

ν_{m^*} local minimizer $\Leftrightarrow f_\nu^-(\nu_{m^*}) < 0$ and $f_\nu^+(\nu_{m^*}) > 0$

$$f_\nu^+(\nu_{m^*}) > 0 \Leftrightarrow f_\nu^-(\nu_{m^*}) + 2w_{m^*} > 0 \Leftrightarrow w_{m^*} > -\frac{1}{2} f_\nu^-(\nu_{m^*})$$

$[\nu_{m^*}, \nu_{m^*+1}]$ local minimizer $\Leftrightarrow f_\nu^-(\nu_{m^*}) < 0, f_\nu^+(\nu_{m^*}) = 0$ and $f_\nu^+(\nu_{m^*+1}) > 0$

$$f_\nu^+(\nu_{m^*}) = 0 \Leftrightarrow f_\nu^-(\nu_{m^*}) + 2w_{m^*} = 0 \Leftrightarrow w_{m^*} = -\frac{1}{2} f_\nu^-(\nu_{m^*})$$

$$f_\nu^+(\nu_{m^*+1}) > 0 \Leftrightarrow \underbrace{f_\nu^+(\nu_{m^*})}_{=0} + 2w_{m^*+1} > 0 \Leftrightarrow w_{m^*+1} > 0$$

□

With these results we can now formulate an algorithm. The idea of the algorithm is as follows: First we check the input data for the conditions of Theorem 2.3 and Theorem 2.4 respectively. If we do not find a solution by this way, we start the following procedure: We check iteratively the derivatives of all existing facilities for finding the local minimizers. As soon as we find a local minimizer, we compare the value of the objective function with the best value for the objective function we found before. We update the objective value if the new value is better than the old one. If two locations with the same objective value exist, we store both of them. With this $O(M \log M)$ procedure we can find all global minimizers.

ALGORITHM 5.1. (Minimization of $f_\nu(x)$)
Input: w_m, ν_m
Output: $\mathcal{X}^*(f_\nu), z_\nu^*$

1. $W := \sum_{m \in \mathcal{M}} w_m$
   if $W < 0 \rightarrow$ stop: the solution is at infinity
   else $\rightarrow$ goto step 2.
2. if a $w_{m^*} > 0$ exists with $w_{m^*} \geq \sum_{\mathcal{M} \setminus \{m^*\}} |w_m| \rightarrow$ stop
   Output: $\mathcal{X}^*(f_\nu) = \{\nu_{m^*}\}$ and $z_\nu^* = f_\nu(\nu_{m^*})$
3. if $\nu_1 < \ldots < \nu_M \rightarrow$ goto step 4.
   else $\rightarrow$ sort the existing facilities and sum up the weights of equal co-ordinates $\rightarrow \tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_{\tilde{M}}), \quad \tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_{\tilde{M}})$ with $\tilde{\nu}_1 < \ldots < \tilde{\nu}_{\tilde{M}}$
4. $f_\nu^-(\tilde{\nu}_1) = -W =: F^-$
   $\mathcal{X}^*(f_\nu) := \emptyset, z_\nu^* := f(\tilde{\nu}_1)$
   $\tilde{\nu}_{\tilde{M}+1} := \infty$

   for $m = 1, \ldots, \tilde{M}$ do
   if $F^- < 0$ and $\tilde{w}_m > -\frac{1}{2}F^-$:
   determine $f(\tilde{\nu}_m)$
   if $f(\tilde{\nu}_m) < z_\nu^*$: $z_\nu^* := f(\tilde{\nu}_m)$, $\mathcal{X}^*(f_\nu) := \{\tilde{\nu}_m\}$
   if $f(\tilde{\nu}_m) = z_\nu^*$: $z_\nu^* := f(\tilde{\nu}_m)$, $\mathcal{X}^*(f_\nu) := \mathcal{X}^*(f_\nu) \cup \{\tilde{\nu}_m\}$
   if $F^- < 0$ and $\tilde{w}_m = -\frac{1}{2}F^-$ and ($\tilde{w}_{m+1} > 0$ or $m = \tilde{M}$):
   determine $f(\tilde{\nu}_m)$
   if $f(\tilde{\nu}_m) < z_\nu^*$: $z_\nu^* := f(\tilde{\nu}_m)$, $\mathcal{X}^*(f_\nu) := \{[\tilde{\nu}_m, \tilde{\nu}_{m+1}]\}$

if $f(\tilde{\nu}_m) = z^*_\nu$: $z^*_\nu := f(\tilde{\nu}_m)$, $\mathcal{X}^*(f_\nu) := \mathcal{X}^*(f_\nu) \cup \{[\tilde{\nu}_m, \tilde{\nu}_{m+1}]\}$

$F^- := F^- + 2\tilde{w}_m$
$m := m + 1$

5. If $\tilde{\nu}_1 \in \mathcal{X}^*(f_\nu)$ and $W = 0$: $\mathcal{X}^*(f_\nu) := \mathcal{X}^*(f_\nu) \cup \{(-\infty, \tilde{\nu}_1]\}$

Output: $\mathcal{X}^*(f_\nu)$ with objective value z^*_ν

To solve the original problem, we use this algorithm for both subproblems and link the two optimal sets.

ALGORITHM 5.2. (Solution of $1/P/w_m \gtrless 0/l_1/\sum$)

1. Use Algorithm 5.1 for $f_a(x_1)$ and $f_b(x_2)$
   $\rightarrow$ $\mathcal{X}^*(f_a)$, $\mathcal{X}^*(f_b)$, z^*_a, z^*_b

Output: $z^* = z^*_a + z^*_b$, $\mathcal{X}^*(f) = \mathcal{X}^*(f_a) \times \mathcal{X}^*(f_b)$

5.2. Solving $1/P/\mathcal{R}, w_m \gtrless 0/l_1/\sum$

As we have seen in the general case in Algorithm 4.2 we need all local minima of the unrestricted case. Therefore, we have to modify Step 4 of Algorithm 5.1 in the following way:

Modification of step 4 of Algorithm 5.1 (Determination of all local optima)
$F^- := -W$
$\mathcal{X}^{loc}(f_\nu) := \emptyset$, $z^{loc}_\nu := \emptyset$
If $W = 0$ and $\tilde{w}_1 > 0$:
$\mathcal{X}^{loc}(f_\nu) := \{(-\infty, \tilde{\nu}_1]\}$
$z^{loc}_\nu := \{f(\tilde{\nu}_1)\}$
$\tilde{\nu}_{\tilde{M}+1} := \infty$
For $m = 1, \ldots, K$ do

if $F^- < 0$ and $\tilde{w}_m > -\frac{1}{2}F^-$:
$\mathcal{X}^{loc}(f_\nu) := \mathcal{X}^{loc}(f_\nu) \cup \{\tilde{\nu}_m\}$, $z^{loc}_\nu := z^{loc}_\nu \cup \{f(\tilde{\nu}_m)\}$
if $F^- < 0$ and $\tilde{w}_m = -\frac{1}{2}F^-$ and ($\tilde{w}_{m+1} > 0$ or $m = \tilde{M}$):
$\mathcal{X}^{loc}(f_\nu) := \mathcal{X}^{loc}(f_\nu) \cup \{[\tilde{\nu}_m, \tilde{\nu}_{m+1}]\}$, $z^{loc}_\nu := z^{loc}_\nu \cup \{f(\tilde{\nu}_m)\}$

$F^- := F^- + 2w_m$
$m = m + 1$
Output: $\mathcal{X}^{loc}(f_\nu)$ set of local minimizer, z^{loc}_ν objective values

With this modification we get the following $O(M^2 \log M)$ algorithm:

ALGORITHM 5.3. (Solution of $1/P/\mathcal{R},\ w_m \gtrless 0/l_1/\sum$)
Input: w_m, Ex_m
Output: $\mathcal{X}^*_{\mathcal{R}}(f)$, $z^*_{\mathcal{R}}$

1. Use Algorithm 5.1 with the modification of step 4 for the two subproblems: We get $\mathcal{X}^{loc}(f_a), \mathcal{X}^{loc}(f_b), z_a^{loc}, z_b^{loc}$ with $|z_a^{loc}| = o_1, |z_b^{loc}| = o_2$
2. For $k = 1, \ldots, o_1, \quad l = 1, \ldots, o_2$ do:
   $z_{kl}^{loc} := z_{a_k}^{loc} + z_{b_l}^{loc}, \quad \mathcal{X}^{loc}(f) := \{(a_k^{loc}, b_l^{loc}, z_{kl}^{loc})\}$
3. Sort $\mathcal{X}^{loc}(f)$ with z_{kl}^{loc} increasing:
   $\rightarrow \widetilde{\mathcal{X}^{loc}}(f)$ with $\tilde{z}_1^{loc} \leq \ldots \leq \tilde{z}_O^{loc} \quad (O = o_1 * o_2)$
4. Determine the best element(s) $\widetilde{X^{loc}}$, which is (are) located in $\mathcal{F}$:
   $\mathcal{X}^*_{\mathcal{R}}(f) := \{(\tilde{a}_{k^*}, \tilde{b}_{l^*})\}, z^*_{\mathcal{R}} := z_{k^*l^*}^{loc}$
5. Determine $\mathcal{Y} := \mathcal{H} \cap \partial\mathcal{R}, |\mathcal{Y}| := L$

6. for $i = 1, \ldots, L$ do:
   determine $f(Y_i)$
   (a) if $f(Y_i) < z^*_{\mathcal{R}}$:
      (i) $z^*_{\mathcal{R}} := f(Y_i)$
      (ii) $\mathcal{X}^*_{\mathcal{R}}(f) := \{Y_i\}$
      (iii) if all optimal solutions should be determined:
         – compute $L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}$
         – $\mathcal{X}^*_{\mathcal{R}}(f) := \{X | X \in L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}\}$
         – If $W = 0$ then compute $L_=(z^*_{\mathcal{R}}) \cap \mathcal{F}$ and $\mathcal{X}^*_{\mathcal{R}}(f) := \{X | X \in L_=(z^*_{\mathcal{R}}) \cap \mathcal{F}\}$
   (b) if $f(Y_i) = z^*_{\mathcal{R}}$
      (i) $\mathcal{X}^*_{\mathcal{R}}(f) := \mathcal{X}^*_{\mathcal{R}}(f) \cup \{Y_i\}$
      (ii) if all optimal solutions should be determined:
         – compute $L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}$
         – $\mathcal{X}^*_{\mathcal{R}}(f) := \mathcal{X}^*_{\mathcal{R}}(f) \cup \{X | X \in L_=(z^*_{\mathcal{R}}) \cap \partial\mathcal{R}\}$
         – if $W = 0$ then compute $L_=(z^*_{\mathcal{R}}) \cap \mathcal{F}$ and $\mathcal{X}^*_{\mathcal{R}}(f) := \{X | X \in L_=(z^*_{\mathcal{R}}) \cap \mathcal{F}\}$

Output: $\mathcal{X}^*_{\mathcal{R}}(f)$ optimal solution, $z^*_{\mathcal{R}}$ objective value

5.3. An illustrative example

We now give an example for the rectangular metric:
$Ex = \{(1,3), (2,1), (4,5), (5,2), (7,3)\}$
$w = (3, 1, -5, -1, 3) \quad \Rightarrow \quad W = 1$
So we get as objective function:

$$\begin{aligned} f(X) = {} & 3\left(|1 - x_1| + |3 - x_2|\right) + \left(|2 - x_1| + |1 - x_2|\right) - 5\left(|4 - x_1| + |5 - x_2|\right) \\ & - \left(|5 - x_1| + |2 - x_2|\right) + 3\left(|7 - x_1| + |3 - x_2|\right) \end{aligned}$$

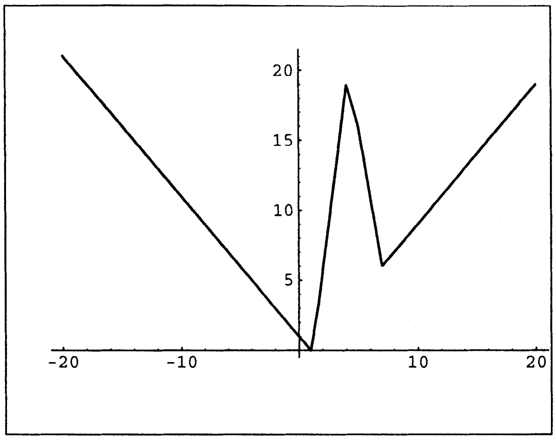

Figure 5.2. function $f_a(x_1)$

$a = (1, 2, 4, 5, 7) = \tilde{a}$ with $w_a = (3, 1, -5, -1, 3) = \tilde{w}_a$
$b = (3, 1, 5, 2, 3) \Rightarrow \tilde{b} = (1, 2, 3, 5)$ with $\tilde{w}_b = (1, -1, 6, -5)$

$f_a(x_1) = 3|1 - x_1| + |2 - x_1| - 5|4 - x_1| - |5 - x_1| + 3|7 - x_1|$
(see Figure 5.2)

	$x_1 < 1$	$1 \le x_1 < 2$	$2 \le x_1 < 4$	$4 \le x_1 < 5$	$5 \le x_1 < 7$	$7 \le x_1$
$p(x_1)$	0	1	2	3	4	5
$f_a(x_1)$	$-x_1 + 1$	$5x_1 - 5$	$7x_1 - 9$	$-3x_1 + 31$	$-5x_1 + 41$	$x_1 - 1$

$f_b(x_2) = 3|3 - x_2| + |1 - x_2| - 5|5 - x_2| - |2 - x_2| + 3|3 - x_2|$
(see Figure 5.3)

	$x_2 < 1$	$1 \le x_2 < 2$	$2 \le x_2 < 3$	$3 \le x_2 < 5$	$5 \le x_2$
$p(x)$	0	1	2	3	4
$f_b(x_2)$	$-x_2 - 8$	$x_2 - 10$	$-x_2 - 6$	$11x_2 - 42$	$x_2 + 8$

First we consider the unrestricted case:
Step 4 of Algorithm 5.1 for $f_a(x_1)$:
Initialization: $F_a^+ := f_a^+(a_1) = 5, \quad \mathcal{X}^*(f_a) := \{1\}, \quad z_a^* := 0$
$m = 2$: $F_a^+ := F_a^+ + 2w_2 = 7$
$m = 3$: $F_a^+ = -3$

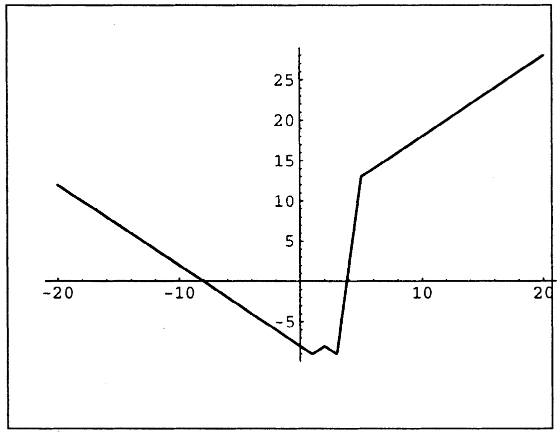

Figure 5.3. function $f_b(x_2)$

$m = 4$: $(-1) < -\frac{1}{2}(-3), \quad F_a^+ = -5$
$m = 5$: $3 > \frac{5}{2} \to f_a(a_5) = f_a(7) = 6 \ (> 0)$
Output: $\mathcal{X}^*(f_a) = \{1\}, \quad z_a^* = 0$
Algorithm 5.1 applied to $f_b(x_2)$ yields: $\mathcal{X}^*(f_b) = \{1, 3\}$ with $z_b^* = -9$

So we get the solution for the original problem with Algorithm 5.2:
$\mathcal{X}^*(f) = \{(1,1), (1,3)\}$ with $z^* = -9$

Next, we introduce a convex forbidden region:
$\mathcal{R}$:=$\{(x_1, x_2) | -6 \le x_1 \le 3, -6 \le x_2 \le 4\}$

Step 1: $\mathcal{X}^{loc}(f_a) = \{1, 7\}$, $z_a^{loc} = \{0, 6\}$
$\mathcal{X}^{loc}(f_b) = \{1, 3\}$, $z_b^{loc} = \{-9, -9\}$

Step 2: $\mathcal{X}^{loc}(f) = \{(1, 1, -9), (1, 3, -9), (7, 1, -3), (7, 3, -3)\}$
Step 3: $\widetilde{\mathcal{X}^{loc}(f)} = \mathcal{X}^{loc}(f)$
Step 4: $\mathcal{X}_{\mathcal{R}}^*(f) = \{(7, 1), (7, 3)\}, \quad z_{\mathcal{R}}^* = (-3)$

Step 5: Intersection points of $\mathcal{H}$ with $\partial\mathcal{R}$
$\mathcal{Y} = \{(-6, 1), (-6, 2), (-6, 3)(1, 4), (2, 4), (3, 3), (3, 2), (3, 1),$
$(2, -6), (1, -6), \}$
$z_{\mathcal{Y}} = \{-2, -1, -2, 2, 7, 3, 4, 3, 3, -2\}$

Output: $\mathcal{X}_{\mathcal{R}}^*(f) = \{(7, 1), (7, 3)\}, z_{\mathcal{R}}^* = (-3)$

Now, the forbidden region is the complement of a closed polygonal region:
$\mathcal{F} = \mathcal{P} := \{(x_1, x_2) | 3 \le x_1 \le 8, 0 \le x_2 \le 4\}$
We get $\mathcal{V}(\mathcal{P}) = \{(3, 0), (3, 4), (8, 4), (8, 0)\}$ with objective values $z_{\mathcal{V}} = \{4, 14, 9, -1\}$

$\mathcal{Y} = \{(3,1),(3,2),(3,3),(4,4),(5,4),(7,4),(8,3),(8,2),(8,1),(7,0),(5,0),(4,0)\}$
$z_{\mathcal{Y}} = \{3,4,3,21,18,8,-2,-1,-2,-2,8,11\}$
The best feasible points are (7,1) and (7,3) with objective value $z^*_{\mathcal{R}} = -3$
Solution: $\mathcal{X}^*_{\mathcal{R}}(f) = \{(7,1),(7,3)\}$ and $z^*_{\mathcal{R}} = -3$

6. Conclusions

We have shown in this paper how global optimization problems which are in general very difficult can be solved exactly for special cases in polynomial time. The methods we used are mainly computational geometry and discretization of continuous problems. The success of the procedures relies heavily on the structure of the level sets. Future research topics are extensions to the multi-facility case and therefore to higher dimensions.

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